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DRAFT
Moments of spin structure functions and QCD corrections

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1 Introduction

This TN is a somewhat naive summary of two related aspects of the moments of the SSF's, namely the low Q^2, high x target mass (TM) corrections to the moments, which result in the Nachtmann moments that mix $g_1$and g_2, and the QCD LO evolution and NLO corrections to the moments. The OPE relates the measured moments to calculable matrix elements, which may have Wilson coefficients representing LO or NLO radiative corrections. The purpose is to clarify the notation used in the literature so the necessary corrections (TM, LO, NLO) can be applied unambiguously to RSS.

2 Matrix elements and Wilson coefficients

Ji and Chou [1] discuss QCD radiative corrections to non-singlet $g_2(x, Q^2)$. Their result reproduces the singlet and non-singlet ones derived by Shuryak and Vainshtein [2]. The notation in Ji and Chou seems more familiar, so we'll follow it initially.

The starting point is their Eqs. (4) and (5), the detailed form of the OPE sum rules

$$\int_0^1 x^n g_1(x, Q^2) dx = \frac{1}{2} \sum_i a_i^n(\mu^2) F_{i,2}^n(Q^2/\mu^2, \alpha_s(\mu^2)) \quad (n = 0, 2, 4, ...) \quad (1)$$

$$\int_0^1 x^n g_2(x, Q^2) dx = \frac{n}{2(n+1)} \Big[\sum_i a_i^n(\mu^2) F_{i,2}^n(Q^2/\mu^2, \alpha_s(\mu^2)) - \sum_i d_i^n(\mu^2) F_{i,3}^n(Q^2/\mu^2, \alpha_s(\mu^2))\Big] \quad (n = 2, 4, 6, ..) \quad (2)$$

The indices i indicate operators of the same Lorentz structure, n is the sum rule or moment order. a_i^n are twist-2 scalar matrix elements, and d_i^n are twist-3 matrix elements. $F^n_{i,2 \text{ or } 3}$ are the coefficient functions or Wilson coefficients that include the logarithmic QCD radiative corrections; their indices 2 or 3 correspond to the twist of the associated matrix elements. μ^2 is the renormalization scale, and the strong coupling constant $\alpha_s(\mu^2) = g(\mu^2)/4\pi$.

For RSS we want the twist-3 part of the moments, which, as usual, can be obtained by substituting the twist-2 moment Eq. (1) in the mixed twist Eq. (2). Then, the twist-3 moment of the spin structure functions is given by

$$\int_0^1 x^n \overline{g_2}(x,Q^2)dx = \frac{n}{2(n+1)} \sum_i d_i^n(\mu^2) F^n_{i,3}(Q^2/\mu^2, \alpha_s(\mu^2)) \qquad (3)$$

We write the left-hand side in terms of the measured structure functions g_1 and g_2

$$\int_0^1 x^n (2g_1(x,Q^2) + 3g_3(x,Q^2))dx = \frac{n}{2(n+1)} \sum_i d_i^n(\mu^2) F^n_{i,3}(Q^2/\mu^2, \alpha_s(\mu^2)) \quad (4)$$

but these are the ordinary Cornwall-Norton moments, which only apply when $Q^2 \gg M^2$ or $M = 0$. At RSS kinematics Q^2=1.28 GeV^2 we need to correct for the nucleon mass, so we need to use the Nachtmann moments [3], which will be discussed in the next section. Here we focus on the matrix elements and the coefficient functions.

The OPE sum rule notation

$$\int_0^1 x^n g_2(x,Q^2)dx = \frac{n}{2(n+1)}(\tilde{d}_n - \tilde{a}_n) \qquad (n = 2,4,6,...) \qquad (5)$$

with $\tilde{d}_n = d_n^{(q)} E^q_{2n} + d_n^{(g)} E^g_{2n}$, and a similar expression for $\tilde{a}_n$, is used by Bass [4] (his Eqs. (47) to (49)) to combine the matrix elements and coefficient functions in a single object, an "effective" matrix element indicated by the tilde. This notation is the one used by other authors, like Dong [5], to relate the "effective" matrix elements used in the simplified expressions for the OPE sum rules to the scalar matrix elements of Ji and Chou, and Matsuda and Uematsu [3].

3 Natchmann moments

When deriving the moments of the spin structure functions for cases when $M^2/Q^2 \sim 1$ the authors of [3] define their moments index $N = n + 1$, so the sum rules they obtain run over *odd* indices (they actually just use n, not upper case N, which adds to the confusion). Their eqs. (18) and (19) are

$$M_1^n(Q^2) \equiv \frac{1}{2} a_n E_1^n(Q^2, g) = \int_0^1 \frac{dx}{x^2} \xi^{n+1} \Big[\Big\{ \frac{x}{\xi} - \frac{n^2}{(n+2)^2} \frac{M^2 x \xi}{Q^2} \Big\} g_1(x, Q^2) - \frac{4n}{n+2} \frac{M^2 x^2}{Q^2} g_2(x, Q^2) \Big] \quad (n = 1, 3, 5, ..) \tag{6}$$

$$M_2^n(Q^2) \equiv \frac{1}{2} d_n E_2^n(Q^2, g) = \int_0^1 \frac{dx}{x^2} \xi^{n+1} \Big[\frac{x}{\xi} g_1(x, Q^2) + \Big\{ \frac{n}{n-1} \frac{x^2}{\xi^2} - \frac{n}{n+1} \frac{M^2 x^2}{Q^2} \Big\} g_2(x, Q^2) \Big] \quad (n = 3, 5, 7, ..) \tag{7}$$

where $\xi = 2x/(1 + \sqrt{1 + (2Mx)^2/Q^2})$ is the Nachtmann scaling variable that corrects for $M^2/Q^2 \sim 1$. Here g is of course just $4\pi\alpha_s(Q^2)$.

The $\tilde{d}_2$ of [5] or [4] corresponds to $d_3 E_2^3$ here and to $\sum_i d_i^2 F_{i,3}^2$ in [1]. The author of [5] seems to confuse the definition of the Nachtmann moments in [3], which has an extra factor of 1/2 compared with the usual OPE definition, with this factor being part of the matrix element. As we will see below, for $n = 2$ (i.e. $N = 3$) there is only one term in the sum ($\sum_i$ ends at $i = 1$), so in this case the equivalence reads

$$\tilde{d}_2 = d_3 E_2^3 = d_1^2 F_{1,3}^2. \tag{8}$$

The corresponding moment in plain $d_2 \equiv \tilde{d}_2$ notation is then

$$d_2(Q^2) = 2M_2^{(3)}(Q^2) = \int_0^1 dx \xi^2 \Big(2\frac{\xi}{x} g_1(x, Q^2) + 3(1 - \frac{\xi^2}{2} \frac{M^2}{Q^2}) g_2(x, Q^2) \Big), \tag{9}$$

which for $M = 0$ or $Q^2 \gg M$ reduces to the C–N one Eq. (4) ($\xi(M^2 = 0) = x$).

3.1 Nachtmann moments of g_1

Using Eq.(6) we can calculate the zeroth moment of the longitudinal SSF

$$M_1^{(1)}(Q^2) \equiv \frac{1}{2} a_1 E_1^1(Q^2, g) =$$
$$\int_0^1 dx \frac{\xi}{x} \Big[\Big\{ 1 - \frac{1}{9} \frac{M^2 \xi^2}{Q^2} \Big\} g_1(x, Q^2) - \frac{4}{3} \frac{M^2 x \xi}{Q^2} g_2(x, Q^2) \Big] \tag{10}$$

This expression obviously reduces to $\Gamma_1(Q^2) = \int_0^1 g_1(x, Q^2) dx$ in the limits $M = 0, M \ll Q^2$.

The advantage of the measurement in RSS of the complete set of SSF's is that the $g_2(x, Q^2)$ results are available to compute the Nachtmann moments, unlike experiments that only measure the parallel asymmetry. Considering that the terms with the M^2/Q^2 factor can contribute up to 20% of the integrands for $Q^2 <\sim 3$ GeV^2 at intermediate and high x, the importance of having data for both parallel and transverse asymmetries cannot be overstated. The need for g_2 grows with x and with the order of the moment.

Since the extended Gerassimov-Drell-Hearn sum rule is based on the first moment of g_1, the existing results for this sum rule at intermediate and low Q^2 need to be recalculated using Nachtmann moments, rather than C–N ones.

For RSS kinematics, the ratio between the C–N and Nachtmann moments is significantly different from unity: the integrals over just the measured region is 0.035±0.001 for C–N, versus 0.0032±0.001 for Nachtmann. This 10% discrepancy (at the three sigmas level) must increase substantially at lower Q^2.

A global analysis of g_1 data has calculated Nachtmann moments [6] using a parameterization of A_2 to estimate g_2. At neighboring values of Q^2 they find $M_1^{(1)}(1.2 \text{ GeV}^2) = 0.0942 \pm 0.0035 \pm 0.0100 \pm 0.0010$ and $M_1^{(1)}(1.4 \text{ GeV}^2) = 0.102 \pm 0.0040 \pm 0.0110 \pm 0.0020$, with statistical, systematic and low x extrapolation errors. These results linearly interpolated to RSS kinematics give $M_1^{(1)}(1.28 \text{ GeV}^2) = 0.0973 \pm 0.0041 \pm 0.0116 \pm 0.0013$ which seems consistent with our C–N result $\Gamma_1^p(1.28 \text{ GeV}^2) = 0.1030 \pm 0.0010 \pm 0.0070$ mainly because of the large systematic errors.

4 d_2 in Leading Order

The application to RSS can be done first in LO by using the result of Eq. (35) in [1], for the case of $n = 2$ (third moment,) for which there is only one operator

$$\int_0^1 x^2 \overline{g}_2^{T=1} dx = F_{1,3}^2(Q^2) d_1^2(\mu^2) = \Big[\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\Big]^{0.95} d_1^2(\mu^2) \tag{11}$$

Here the non-singlet twist-3 part of g_2 is $\overline{g}_2^{T=1} = \overline{g}_2^p - \overline{g}_2^n$. The exponent is the anomalous dimension $\gamma_{1,i=1}^{n=2}/(2\beta) = (3C_A - \frac{1}{3}C_F)/\beta_0 = 0.95$ (for $N_F = 3$) of the non-singlet coefficient function

$$F_{1,3}^2(Q^2) = \Big[\frac{\alpha_s(\mu^2)}{\alpha_s(Q^2)}\Big]^{-\gamma_{11}^2/(2\beta)} F_{1,3}^2(\mu^2) \tag{12}$$

with $C_A = 3, C_F = (N_c^2 - 1)/(2N_c)$, the number of colors $N_c = 3$ and $\beta_0 = 9$, which is identical to the corresponding anomalous dimension found in [2]. To get eq. (11), $F_{13}^2(1,0) = 1$.

Obviously, for $\mu^2 = Q^2$ there is no LO correction so this result only helps with comparing measurements of $d_2(Q^2)$ at different values of Q^2. I think this is what Ji *et al.* [9] mean when they indicate that d_2 receives no radiative correction, i.e. in LO. Therefore, for RSS in LO, our d_2 results based on Nachtmann moments are it.

The corresponding anomalous dimensions for the singlet $T = 0$ part of $\overline{g}_2^{T=0} = \overline{g}_2^p + \overline{g}_2^n$ is found in [2]. The result is

$$\gamma^\Sigma = \frac{3N_c - \frac{1}{6}(N_c - \frac{1}{N_c}) + \frac{2}{3}N_F}{\beta_0}. \tag{13}$$

The evolution of the singlet moment (for $N_F = 3$) is

$$\int_0^1 x^2 \overline{g}_2^{T=0} dx = F_{1,3}^2(Q^2) d_1^2(\mu^2) = \Big[\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\Big]^{1.17} d_1^2(\mu^2) \tag{14}$$

As for the NS case $F_{13}^2(1,0) = 1$.

4.1 Flavor structure of the moments

In QCD, the NS and Σ parts of structure functions are dependent on the number of flavors N_F. They can be related to SU(2) isospin symmetry only for two active quark flavors (u and d) [7]. Otherwise, SU(3) or higher symmetries apply and, therefore, proton and neutron (or deuteron) data alone aren't sufficient to relate the NS and Σ contributions. This can be seen in the Bjorken and Ellis-Jaffe sum rules, that could not be used to extract the quark spin of the nucleon (the a_0 matrix element) without either assuming that the strange quark polarization Δs was zero (Ellis-Jaffe) or resorting to flavor SU(3) symmetry, and getting Δs from the decays of the baryon octet [8].

It would then seem that the NLO corrections may be difficult to apply to d_2, since there are no analogs to SU(3) flavor for g_2. But the x^2 weight in the moment emphasizes the high x, or valence quarks, region, which is dominated by just the u and d flavors. The contributions of the s quark to g_2^s are suppressed by the very small or negligible $s(x)$ quark density at high x. Therefore, unlike the first moments, which have significant contributions from low x, where the sea and heavy quarks are important, the corrections to the d_2 integral can be treated in terms of only the two light flavors. So one can write

$$\begin{aligned} d_2^{NS} &= d_2^p - d_2^n \\ d_2^{\Sigma} &= d_2^p + d_2^n = d_2^d/(1-1.5w_D) \end{aligned} \tag{15}$$

where $w_D \simeq 0.05$ takes into account the D-state correction to the effective nucleon polarization in the deuteron.

4.2 Example: the evolution of the SLAC $d_2^{p,n}(Q^2 = 5\ \mathrm{GeV}^2)$ measurements

We solve the singlet (Σ) and non-singlet (NS) moments of the twist-3 part of g_2 for their p and n components. As we discussed above, in the high x region that dominates the third moments the singlet and non-singlet parts can be related to the $T=0$ and $T=1$ isospin combinations, respectively. We have

$$\begin{aligned} d_2^p(Q^2) &= \frac{1}{2}(d_2^{\Sigma}(Q^2) + d_2^{NS}(Q^2)) \\ &= \frac{1}{2}\Big[d_2^{\Sigma}(\mu^2)\Big(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\Big)^{\Sigma} + d_2^{NS}(\mu^2)\Big(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\Big)^{NS}\Big] \end{aligned} \tag{16}$$

where Σ and NS are shorthand for the corresponding anomalous dimensions. The $N_F = 3$ anomalous dimensions were given above. The $N_F = 4$ anomalous dimensions are $\gamma_{\Sigma} = 1.347$ and $\gamma_{NS} = 1.027$.

The ratios $\alpha_s(Q^2)/\alpha_s(\mu^2)$ reduce to $\ln(\mu^2/\Lambda^2)/\ln(Q^2/\Lambda^2)$, and taking $\mu^2 = 5$ GeV^2 for the SLAC measurements [11] $d_2^p = 0.0032 \pm 0.0017$ and

$d_2^n = 0.0079 \pm 0.0048$, the singlet $d_2^\Sigma = 0.0111 \pm 0.0045$ and non-singlet $d_2^{NS} = -0.0047 \pm 0.0045$, where the partial correlation of the proton and neutron errors is taken into account, since both the singlet and neutron quantities come from the same proton and deuteron results, etc. Then, the SLAC result evolved down to the RSS value 1.28 GeV^2 is $d_2^p(1.28\ \text{GeV}^2) = 0.0057 \pm 0.0030$, with the same relative error as the unevolved d_2^p. This is to be compared with our Cornwall-Norton result $0.0057 \pm 0.0009 \pm 0.0007$.

The singlet moment can also be obtained directly from the deuteron d_2^d results using Eq. (15). Combining deuteron d_2 from E143 [12], E155 [13] and E155x [11], we get $d_2^\Sigma = 0.0126 \pm 0.0044$ and the evolved $d_2^p(1.28\ \text{GeV}^2) = 0.0059 \pm 0.0031$. To get the result at 5 GeV^2, the published SLAC per-nucleon d_2^d were multiplied by a factor of 2.

To compare the Nachtmann RSS result for d_2 to the evolved SLAC result, we would need to apply the TM correction of [5] to the SLAC data. Instead, a better approach is to take advantage of RSS's smaller errors than SLAC's and evolve RSS Nachtmann's result $d_2^p(1.28\ \text{GeV}^2) = 0.0036 \pm 0.0006 \pm 0.0004$ up to SLAC's Q^2, to compare with their Nachtmann-corrected result. The result is $d_2^p(5\ \text{GeV}^2) = 0.0021 \pm 0.0004$ (total error), compared to the Nachtmann corrected SLAC result [5] $d_2^p(5\ \text{GeV}^2) = 0.0028 \pm 0.0015$ (total error).

We can also make a significant comparison between RSS Nachtmann singlet $d_2^\Sigma(1.28\ \text{GeV}^2) = 0.0052 \pm 0.0022$ and SLAC's C–N result $d_2^\Sigma = 0.0126 \pm 0.0044$, with Nachtmann TM corrections from ref. [5], combined to form a TM corrected singlet $d_2^\Sigma = (0.86 + 0.90) d_2^{d(C-N)}/2 = 0.0111 \pm 0.0039$. The corresponding RSS Nachtmann $d_2^\Sigma(5\ \text{GeV}^2) = 0.0029 \pm 0.0012$ is more than six RSS sigmas away, inconsistent with the SLAC result even considering all measurement errors. In contrast, the RSS C–N singlet moment $d_2^\Sigma(1.28\ \text{GeV}^2) = 0.0089 \pm 0.0037$ evolved to SLAC's momentum transfer gives $d_2^\Sigma(5\ \text{GeV}^2) = 0.0049 \pm 0.0020$. The RSS and SLAC C–N singlets are significantly closer to each other than the Nachtmann ones.

In passing, it is interesting to notice that SLAC's deuteron measurement has better relative errors than their proton one and than RSS's deuteron.

4.3 Low x contribution to RSS d_2^p

Our published g_2^p results show that at low x g_2^p is dominated by the twist-2 contribution, within statistical errors. We cannot use the twist-2 part of g_2 to estimate the unmeasured DIS contribution to d_2 because d_2 is pure twist-3. We can only estimate the error of the twist-3 contribution for $x < x_{min}$

under the assumption that the DIS twist-3 is zero, as indicated by our low x results. Plots of the error bands of the A_1 and A_2 fits, suitably converted to fir error bands for g_2 and g_2^{WW}, are shown on Fig. 1 along with the measured data. It is clear that g_2 and g_2^{WW} are equal within errors below $x_{min} = 0.371$ (tabulated value), confirming our statement that there is negligible twist-3 in our measured low x region. From the properties of higher twists we would expect this to be true down to $x = 0$. A tendency towards a constant trend consistent with zero $\overline{g}_2^p$ can be seen on Fig. 2.

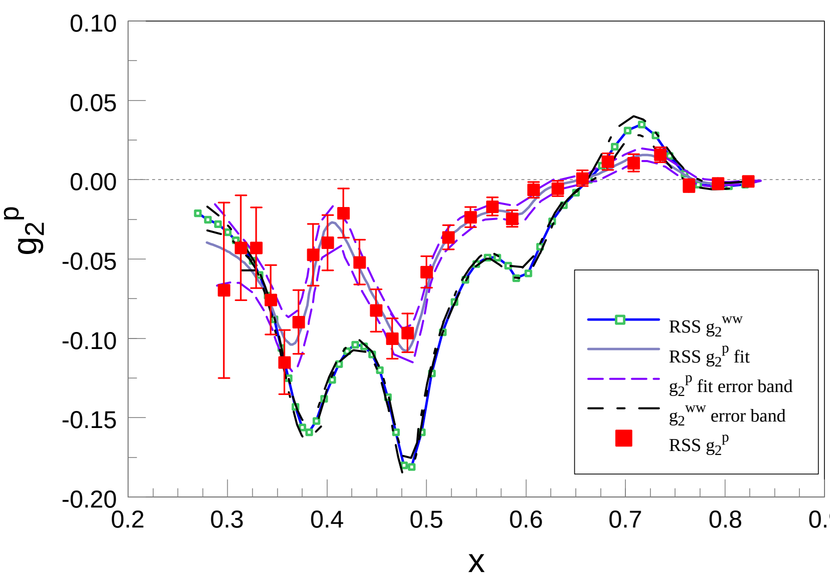

Figure 1: RSS results and fits with error bands for g_2 and g_2^{WW}. The error band for g_2^{WW} is narrow due to a combination of smaller fit errors for g_1 than for g_2 and correlations between the two terms of $g_2^{WW} = -g_1 + \int_x^1 dy g_1(y)/y$.

Since the low x region is rather insensitive to the target mass correction, as illustrated by the near-unity ratio of the scaling variables $\xi/x = 0.94$ at RSS lowest $x = 0.316$, calculated at our average $\langle Q^2 \rangle = 1.279$ GeV^2, it

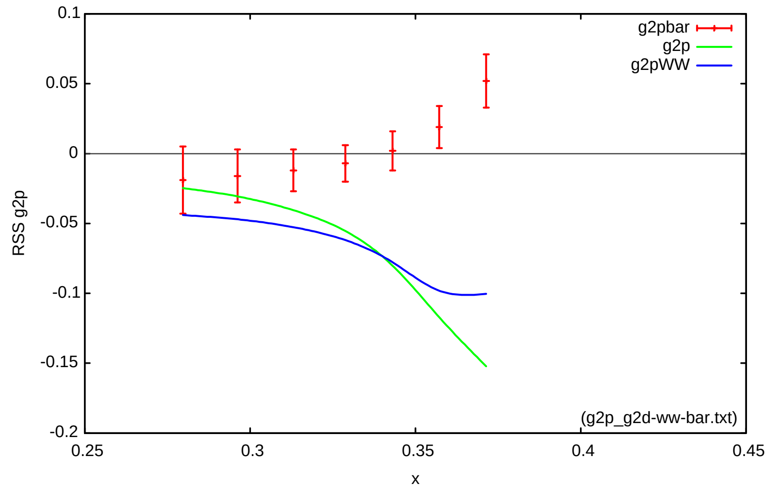

Figure 2: RSS fits to g_2 and g_2^{WW} and errors for $\overline{g}_2^p$ in the region near x_{min}.

should be acceptable to use the simpler Cornwall-Norton moments to write

$$d_2 = \int_0^{x_{min}} 3x^2\overline{g_2}dx + \int_{x_{min}}^1 x^2(2g_1 + 3g_2)dx = d_2^{DIS} + d_2^{Resonances} \tag{17}$$

where we take $d_2^{DIS} = 0$ and $d_2^{Resonances} = \overline{d_2^{RSS}}$.

The error on d_2^{DIS} can be determined from the fit error

$$\begin{aligned}
\delta d_2^{DIS} &= d_2^{DIS}(\overline{g_2} + \delta\overline{g_2}) - d_2^{DIS}(\overline{g_2}) = 3\int_0^{x_{min}} x^2(\delta\overline{g_2})^2dx \\
\delta d_2^{DIS}(Q^2) &= x_{min}^3\sqrt{(\delta g_2(x_{min}, Q^2))^2 + (\delta g_2^{WW}(x_{min}, Q^2))^2}
\end{aligned} \tag{18}$$

where we assumed that the errors in g_2 and g_2^{WW} are independent of x over the DIS range, and equal to their fit error values at $x_{min}(\langle Q^2\rangle)$. With $\delta g_2(x_{min}, \langle Q^2\rangle) = 0.02263$ and $\delta g_2^{WW}(x_{min}, \langle Q^2\rangle) = 0.01056$ the resulting contribution of the DIS region to the error in d_2 is $\delta d_2^{DIS} = 0.00079$.

An alternative choice for the error in δd_2^{DIS} would be to average the values of the fit errors in the x region where $\overline{g_2} = 0$, as shown on the plot. This approach is less conservative than taking the fit errors at x_{min} but it is less sensitive to the fit behavior at the end points. The result, averaged over $0.316 \leq x \leq 0.371$, is $\langle\delta d_2^{DIS}\rangle = 0.00060$.

The complete result for the proton is then

$$
\begin{aligned}
d_2^p &= 0.00365 \pm 0.0006(\text{stat.}) \pm 0.0004(\text{syst.}) \pm 0.0008(\text{DIS}) \\
&= 0.00365 \pm 0.0011(\text{total error}). \qquad (19)
\end{aligned}
$$

evidence of twist-3 to better than 3.3 sigmas.

4.4 Deuteron d_2 at low x.

The corresponding data and fits for the deuteron g_2 are shown on Fig. 3. In the region near x_{min} shown on Fig. 4 the twist-3 $\overline{g}_2^d$ seems to be constant but marginally above zero. A linear fit $\overline{g}_2^d = mx + b$, with parameters $m = -0.081 \pm 0.42$, $b = 0.06 \pm 0.14$, and a constant $\overline{g}_2^d = 0.0339 \pm 0.011$ fit are shown.

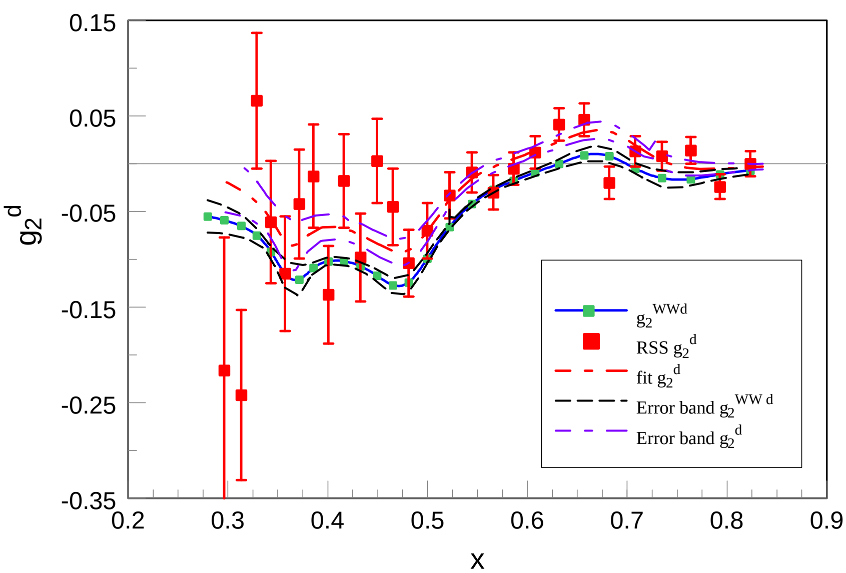

Figure 3: RSS results and fits with error bands for g_2 and g_2^{WW} for the deuteron, as on Fig. 1.

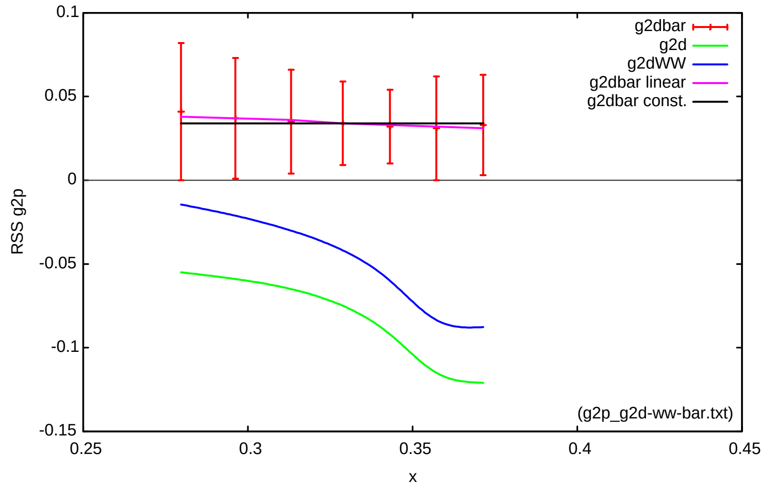

Figure 4: RSS fits to g_2 and g_2^{WW} and errors for $\overline{g}_2^d$ in the region near x_{min}.

Assuming that the observed dependence of $\overline{g}_2^d$ near x_{min} can be extrapolated to zero, the integral for the linear fit is $d_2^{d\ DIS} = 0.00132$ and for the constant fits to $\overline{g}_2^d = 0.00107$. Their average is $\langle d_2^{d\ DIS} \rangle = 0.00119 \pm 0.00125$, with the error estimated following the same reasoning used for the proton, (Eq. 18), with the addition of a "model" error equal to one-half the difference between the linear and constant extrapolations of $\overline{g}_2^d$.

Combining the measured Nachtmann moment for $d_2^{d\ Resonances} = 0.0048 \pm 0.0021$, including systematic and statistical errors, with the DIS contribution the result for the deuteron is

$$\begin{aligned} d_2^d(\langle Q^2 \rangle = 1.28\mathrm{GeV}^2) &= 0.0060 \pm 0.0020(\mathrm{stat.}) \pm 0.0004(\mathrm{syst.}) \pm 0.0012(\mathrm{DIS}) \\ &= 0.0060 \pm 0.0024(\mathrm{total\ error}), \end{aligned} \tag{20}$$

indicating the presence of twist-3 to better than 2.5 sigmas.

4.5 The singlet and non-singlet d_2

With the definitions of Eq. (15), the singlet d_2^Σ and non-singlet d_2^{NS} twist-3 matrix elements based on Nachtmann moments are

$$\begin{aligned} d_2^\Sigma &= \frac{d_2^d}{0.925} = 0.0065 \pm 0.0026, \\ d_2^{NS} &= 2d_2^p - d_2^\Sigma = 0.0008 \pm 0.0034, \end{aligned}$$

where the errors are total errors. The non-singlet twist-3 matrix element is entirely consistent with zero.

5 d_2^p in Next-to-Leading Order

The NLO corrections to d_2 [9, 10] are again given in terms of non-singlet (NS) and singlet (Σ) parts. With these corrections we can evolve our low Q^2 result even more accurately to SLAC's and higher momentum transfers.

The relevant expression for the non-singlet NLO correction for the twist-3 part of g_2 is (eq.(28) of [9])

$$\begin{aligned}\int_0^1 x^2\Big(3g_T^{NS} - g_1^{NS}\Big)dx &= \sum_i \hat{e}_i^2 d_{2i}\big[1 + \frac{\alpha_s(Q^2)}{4\pi}\big(\frac{27}{4}C_A - \frac{29}{3}C_F\big)\big] \\ \int_0^1 x^2\Big(2g_1^{NS} + 3g_2^{NS}\Big)dx &= \sum_i \hat{e}_i^2 d_{2i} F_3^{2NS} \end{aligned} \qquad (21)$$

where the second line is our familiar form for the (C–N) third moment on the left hand side and the F_3^{2NS} is the Wilson coefficient for the $n = 2$, twist-3 NS moment. $\hat{e}_i^2 = e_i^2 - \overline{e}^2 = e_i^2 - e_i^2/N_F$ is the NS part of the squared quark charges for N_F flavors.

Similarly, for the singlet NLO correction we have (eq.(23) of [10])

$$\begin{aligned}\int_0^1 x^2\Big(3g_T^{NS} - g_1^{NS}\Big)dx &= \overline{e}^2\ d_2^{\Sigma}\big[1 + \frac{\alpha_s(Q^2)}{4\pi}\big(\frac{27}{4}C_A - \frac{29}{3}C_F + \frac{10}{3}N_F T_F\big)\big] \\ \int_0^1 x^2\Big(2g_1^{NS} + 3g_2^{NS}\Big)dx &= \overline{e}^2\ d_2^{\Sigma} F_3^{2\Sigma} = \overline{e}^2\ d_2^{\Sigma}(F_3^{2NS} + \frac{\alpha_s}{4\pi}\frac{10}{3}N_F T_F) \end{aligned} \qquad (22)$$

using the notation defined for Eq.(21), and $T_F = 1/2$ is called the generator normalization.

Assuming that the non-singlet and singlet pieces of d_2 can be combined to form $d_2^{p,n}$ as before, we have

$$\begin{aligned}\int_0^1 x^2\Big(2g_1^{p,n} + 3g_2^{p,n}\Big)dx &= d_2^{p,n}F_3^{2NS} + (-)\frac{1}{2}\overline{e}^2 d_2^{\Sigma}\frac{\alpha_s(Q^2)}{4\pi}\frac{10}{3}N_F T_F \\ &= d_2^{p,n}F_3^{2NS} + (-)\frac{5}{18}d_2^{\Sigma}\frac{\alpha_s(Q^2)}{4\pi}\frac{10}{3}T_F \\ &= d_2^{p,n}F_3^{2NS} + (-)\frac{25}{54}d_2^{\Sigma}\frac{\alpha_s(Q^2)}{4\pi} \end{aligned} \qquad (23)$$

where $\overline{e}^2$ has been calculated for u and d quarks only (but note that the number of flavors N_F cancels out anyway) and the $(-)$ sign applies to the corrections to the neutron moments. d_2^{Σ} is related to d_2^d according to Eq.(15).

The constant factors in F_3^{2NS} equal $265/144/\pi = 1.8402/\pi$. At RSS $\alpha_s(1.28 \text{ GeV}^2) = 0.4432$, so the corrections to the measured proton Nachtmann moment $2M_2^{(3)}$ would be

$$d_2^p = \frac{1}{1+0.26}\Big(2M_2^{(3)} - 0.016\frac{d_2^d}{0.925}\Big). \tag{24}$$

The singlet contribution is negligible, since $d_2^p \sim d_2^d$, so the overall correction is about 0.79. The corresponding correction to the SLAC data would be 0.95 for $\alpha_s(5 \text{ GeV}^2) = 0.2847$, so at SLAC's kinematics our measurement would be $d_2 = 0.0019$ compared to SLAC's NLO $d_2 = 0.0025$ (with TM corrections).

To avoid the complication of combining the NS and Σ squared quark charges $\hat{e}_i^2$ and $\overline{e}^2$, the NS and singlet parts of the measured moments can be computed directly and the NLO corrections can be applied to them instead. Then, $d_2^{p,n}$ can be obtained using Eq.(15). The Nachtmann measured results for RSS are

$$\begin{aligned} 2M_2^{(3)NS} = 2M_2^{(3)p} - 2M_2^{(3)n} &= 4M_2^{(3)p} - \frac{2M_2^{(3)d}}{0.925} \\ &= 2\times 0.0036 - \frac{0.0048}{0.925} = 0.0021 \quad (25) \\ 2M_2^{(3)\Sigma} = \frac{2M_2^{(3)d}}{0.925} &= 0.0052 \quad (26) \end{aligned}$$

The respective NLO corrections are $F_3^{2NS} = 1.26$ and $F_3^{2\Sigma} = 1.436$, which result in

$$\overline{d}_2{}^p(1.28\text{GeV}^2) = \frac{1}{2}\Big(\frac{\overline{d}_2{}^{NS}}{1.26} + \frac{\overline{d}_2{}^{\Sigma}}{1.436}\Big) = 0.0028 \pm 0.0006 \tag{27}$$

where the error represents combined statistical and systematic measured errors. The full NLO corrected matrix element can be calculated using the results of Sec.(4.5)

$$d_2^p(1.28\text{GeV}^2) = \frac{1}{2}\Big(\frac{d_2^{NS}}{1.26} + \frac{d_2^{\Sigma}}{1.436}\Big) = 0.0026 \pm 0.0005 \pm 0.0006 \tag{28}$$

where the first error represents combined measured errors and the second error comes from the low x contribution.

The NLO corrected NS and singlet pieces can be evolved to SLAC's Q^2 as in section 3.2. The exact evolved RSS result for $d_2^p = 0.0019 \pm 0.0004$ following this approach is entirely consistent with the first method, which neglected the small d_2^Σ NLO correction.

The proton results are summarized on Fig. 5.

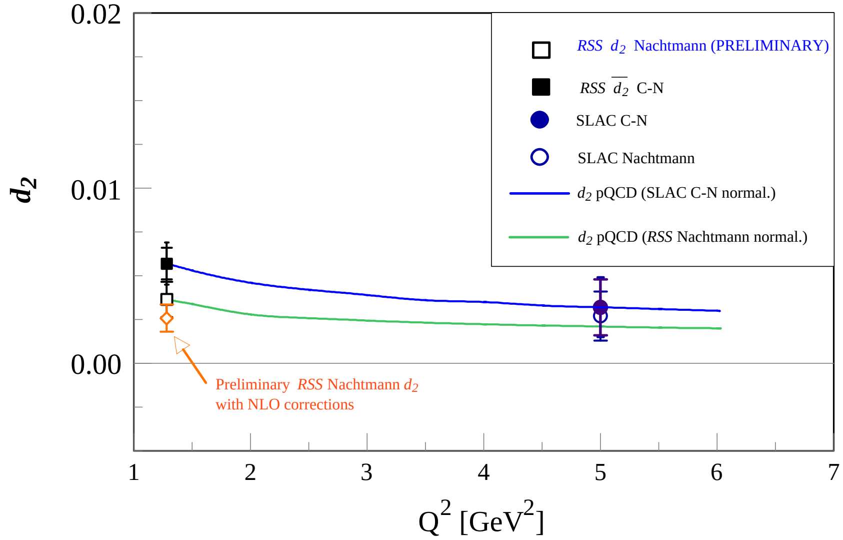

Figure 5: RSS result for d_2^p computed from Cornwall-Norton moments (solid square) and from Nachtmann moments (open square), with NLO corrections (open diamond), plotted along with the expected pQCD evolution [2, 1] and the SLAC C-N results [11] (solid circle) and with Nachtmann TM corrections [5] (open circle).

APPENDIX: The Strong Coupling

The value of α_S is only crudely approximated by the LO expression for $\alpha_s(Q^2)$ [14]

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0 \ln(Q^2/\Lambda^2)} = \frac{12\pi}{(33-2N_F)\ln(Q^2/\Lambda^2)} \tag{29}$$

with LO $\beta_0 = (33-2N_F)/3$, the number of flavors N_F and the QCD parameter is Λ.

For actual application at the scale $\mu = 1.131$GeV, corresponding to RSS $Q^2 = 1.28$ GeV^2, accurate values of α_S can be obtained using the web tool at `http://www-theory.lbl.gov/ ianh/alpha/alpha.html`.

The tool is based on Eq. (9.5) in the section on QCD of the Review of Particle Physics [15], which can be used in conjunction with values of $\Lambda_{QCD}^{(NF)}$ appropriate to the mass scale under study. Using the web tool's results Eq. (9.5) can be solved numerically to obtain $\Lambda^{(3)} = 0.349607$ and $\Lambda^{(4)} = 0.288105$ ($\Lambda^{(5)} = 0.217$ is given in section 9.13 of [15]). Since the c quark mass is $\simeq 1.3$ GeV and $m_s < 0.130$ GeV, the mass scale of RSS corresponds to three flavors, SLAC's to four, and five flavors contribute at the M_Z scale.

To illustrate the difference between the LO formula and the higher order corrections, at LO $\alpha_s(RSS) = 0.5945$, while the higher order result is 0.4432 (used here). A value of $\Lambda = 0.234$ is needed to get the LO expression agree with the higher order result, but this value of Λ leads to α_S(1.2999 GeV)=0.307, off by 30% from the correct 0.396.

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